TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 351, Number 10, Pages 4069–4088
S 0002-9947(99)02175-3
Article electronically published on July 1, 1999

INVERSE EIGENVALUE PROBLEMS ON DIRECTED GRAPHS

ROBERT CARLSON

Abstract. The differential operators iD and $-D^2 - p$ are constructed on certain finite directed weighted graphs. Two types of inverse spectral problems are considered. First, information about the graph weights and boundary conditions is extracted from the spectrum of $-D^2$. Second, the compactness of isospectral sets for $-D^2 - p$ is established by computation of the residues of the zeta function.

1. Introduction

This work contains a number of inverse spectral results for differential operators on graphs. There are a number of physical settings where one might wish to consider differential operators on a graph: heat flow in a wire mesh, mechanical vibrations of networks of elastic strings, propagation of radiation in networks of optical fibers, and electron flow in quantum mechanical circuits. With a few recent exceptions [9, 10], the main mathematical studies relating operator theory and graphs are limited to difference operators [3, 4, 6, 15, 19].

The point of view here is that a differential operator on a graph is like a differential operator on a one dimensional manifold. In addition to the combinatorial structure of the graph, weights are assigned to the edges to provide a metric structure. In order to have a well defined first derivative, the graph is directed. Of course the vertices are exceptional points in our topological space. Here boundary conditions are used to define the domain of the operator.

In this work we are interested in considering inverse eigenvalue problems for the second derivative and Schrödinger operators on graphs. These inverse problems have rich theories in both the classical one dimensional case and in the manifold context. Moreover, even in the one dimensional setting such basic results as compactness of isospectral classes depend on a suitable choice of boundary conditions. One of the main aims of this work is to generalize the isospectral compactness results for one dimensional periodic problems. As a consequence, we have chosen boundary conditions leading to a skew adjoint first derivative operator.

The next section contains a precise description of the operator iD for certain directed graphs with weighted edges and boundary conditions at the vertices. With the appropriate domain, the operator iD will be self adjoint if the conditions at the vertices are given by unitary matrices.

The third section contains inverse spectral results for the operator $-D^2$, which may be roughly described as follows. Typically, all eigenvalues of $-D^2$ are simple.

Received by the editors May 13, 1996 and, in revised form, April 7, 1997.
1991 *Mathematics Subject Classification.* Primary 34L05.
Key words and phrases. Inverse eigenvalue problem, graph spectral theory, zeta function.

If the eigenvalues and the edge weights are known, then the real parts of the eigenvalues of the boundary matrix may be determined. For rather trivial reasons the weights may not be determined from the eigenvalues. What may be determined typically is the sum of the edge weights for loops in the graph. These may be interpreted as the lengths of the loops, and the relationship between the eigenvalues of $-D^2$ and the lengths of the loops may be viewed as an analog of results relating the spectrum of the Laplacian on a manifold to the lengths of closed geodesics [7, p. 170].

The fourth section considers the zeta function ζ_p of $-D^2 - p$. Within our class of operators, the singular parts of the zeta function do not depend on the boundary conditions. For the purposes of studying the residues of ζ_p, this reduces the problem to a diagonal system of operators on circles. The standard theory then implies that isospectral classes of Schrödinger operators are compact.

Before beginning, we mention some notational conventions. The derivative with respect to x is denoted ∂_x. The real and complex numbers are $\mathcal{R}$ and $\mathcal{C}$ respectively. The Hilbert space inner product is $\langle , \rangle$.

It a pleasure to thank M. Harmer, B. Pavlov and the referee for helpful comments.

2. Directed graphs and the operator iD

There is a fruitful interplay between certain graphs and differential operators. The graph $\mathcal{G}$ is assumed to be finite and directed. Multiple edges between vertices are allowed. Each vertex v has $\delta(v) > 0$ input (entering) edges and an equal number of output (exiting) edges. The edges, denoted e_n, $n = 1, \dots, N$, have weights $w_n > 0$. In addition, each vertex v has an invertible linear transformation

$$T(v) : \mathcal{C}^{\delta(v)} \to \mathcal{C}^{\delta(v)}$$

which will be used in describing boundary conditions.

Thinking of a directed edge e_n as an interval $[0, w_n]$, we consider differential operators which act by $f \to if'(x)$ for functions in their domain, which will be a subset of $\bigoplus_n L^2[0, w_n]$. While this description facilitates geometric thinking, computations are simplified by making a linear change of variables so that each edge becomes $[0, 1]$. Our Hilbert space becomes the weighted space $\bigoplus_n L^2([0, 1], w_n)$ with inner product

$$\langle f, g \rangle = \sum_n \int_0^1 f_n(x)\overline{g_n(x)}\ w_n\ dx, \quad f = \begin{pmatrix} f_1 \\ \vdots \\ f_N \end{pmatrix}, \quad g = \begin{pmatrix} g_1 \\ \vdots \\ g_N \end{pmatrix}.$$

On its domain, the operator iD will act by

$$\begin{pmatrix} f_1 \\ \vdots \\ f_N \end{pmatrix} \to i \begin{pmatrix} \partial_x f_1 / w_1 \\ \vdots \\ \partial_x f_N / w_N \end{pmatrix}.$$

The local coordinate mapping of edges to intervals is chosen so that edges exit a vertex at 0 and enter at 1. The domain of iD will be determined by boundary conditions at the vertices. Let $F_i(v)$ be the $\delta(v) - tuple$ of input values of f at v:

$$F_i(v) = \begin{pmatrix} f_{n(1)}(1) \\ \vdots \\ f_{n(\delta(v))}(1) \end{pmatrix}.$$

The $\delta(v) - tuple$ of corresponding output values will be $F_o(v)$. An ordering of the inputs and outputs is provided by the ordering of the edges. Functions in the domain of iD will be required to satisfy the vertex conditions

$$F_i(v) = T(v)F_o(v). \tag{2.a}$$

Perhaps the simplest way to initiate the study of iD as a Hilbert space operator is to start with the maximal operator iD_{max}, whose domain consists of all $f : [0,1] \to \mathcal{C}^N$ with absolutely continuous components and whose derivatives are in $L^2[0,1]$. An easy extension of standard results shows that iD_{max} is a Fredholm operator of index N. The operator iD has domain consisting of those functions in the domain of iD_{max} which satisfy the boundary conditions (2.a). (See for instance [12, pp. 145,169,188,272].)

Let $\langle F, G\rangle_v$ denote the usual complex inner product on $\mathcal{C}^{\delta(v)}$. Then for functions f, g in the domain of iD

$$\begin{aligned} \langle iDf, g\rangle - \langle f, iDg\rangle &= \sum_n \int_0^1 \Big[\frac{i}{w_n}(\partial_x f_n)\overline{g}_n - f_n \overline{\frac{i}{w_n}\partial_x g_n}\Big]\ w_n\ dx \\ &= i\sum_v [\langle F_i(v), G_i(v)\rangle_v - \langle F_o(v), G_o(v)\rangle_v] \\ &= i\sum_v [\langle T(v)F_o(v), T(v)G_o(v)\rangle_v - \langle F_o(v), G_o(v)\rangle_v] \\ &= i\sum_v \langle (T^*(v)T(v) - I)F_o(v), G_o(v)\rangle_v. \end{aligned} \tag{2.b}$$

Here $T^*(v)$ denotes the conjugate transpose of $T(v)$.

Since the domain of iD is defined by N boundary conditions, iD is Fredholm with index at least 0. Suppose that $T^*(v)T(v) - I = 0$ for all vertices v. The computation (2.b) shows that iD has no eigenvalues with nonzero imaginary part, so the index is 0 and iD is self adjoint. Henceforth the matrices $T(v)$ are assumed to be unitary.

This discussion is summarized in the following result.

Proposition 2.1. *Suppose that the matrix $T(v)$ is unitary for each vertex $v \in \mathcal{G}$. Let iD be the operator*

$$\begin{pmatrix} f_1 \\ \vdots \\ f_N \end{pmatrix} \to i \begin{pmatrix} \partial_x f_1/w_1 \\ \vdots \\ \partial_x f_N/w_N \end{pmatrix}$$

on $\bigoplus_n L^2([0,1], w_n)$, whose domain consists of all $f : [0,1] \to \mathcal{C}^N$ whose components are absolutely continuous with derivatives in $L^2[0,1]$ and which satisfy the boundary conditions (2.a). *Then iD is self adjoint.*

It will be convenient to let W denote the diagonal matrix $\mathrm{diag}[w_n]$. Any eigenfunction for $\pm iD$ must satisfy the system of equations

$$\pm iW^{-1}\partial_x Y = \lambda Y, \quad Y(x,\lambda) = \begin{pmatrix} y_1(x,\lambda) \\ \vdots \\ y_N(x,\lambda) \end{pmatrix}. \tag{2.c}$$

The solution matrix

$$\mathbf{Y} = \exp(\mp iW\lambda x)$$

has columns forming a basis of solutions to (2.c). The conditions (2.a) defining the domain of iD may be written as a single linear system

$$f_m(1) - \sum_{n=1}^{N} T_{mn} f_n(0) = 0, \quad m = 1, \dots, N,$$

and λ will be an eigenvalue of $\pm iD$ exactly when the condition

$$\det[\mathbf{Y}(1,\lambda) - T\mathbf{Y}(0,\lambda)] = \det[\exp(\mp iW\lambda) - T] = 0, \quad T = (T_{mn}), \tag{2.d}$$

is satisfied.

An explicit description of the resolvent

$$R_1(\lambda) = [iD - \lambda I]^{-1}$$

will be useful. Solutions of

$$iW^{-1}\partial_x Y - \lambda Y = f, \quad Y = \begin{pmatrix} y_1 \\ \vdots \\ y_N \end{pmatrix}, \quad f = \begin{pmatrix} f_1 \\ \vdots \\ f_N \end{pmatrix}$$

have the form

$$Y(x,\lambda) = \exp(-i\lambda x W)K(\lambda) - i\int_0^x \exp(i\lambda[t-x]W)Wf(t)\ dt, \quad K(\lambda) \in C^N.$$

Satisfaction of the boundary conditions $TY(0) - Y(1) = 0$ requires

$$K(\lambda) = i[\exp(-i\lambda W) - T]^{-1}\int_0^1 \exp(i\lambda[t-1]W)Wf(t)\ dt.$$

For notational convenience let

$$M(\lambda) = [I - \exp(i\lambda W)T]^{-1}.$$

Thus

$$\begin{aligned}[R_1(\lambda)f](x) = i\exp(-i\lambda xW)M(\lambda)\int_0^1 \exp(i\lambda tW)Wf(t)\ dt \\ - i\int_0^x \exp(i\lambda[t-x]W)Wf(t)\ dt.\end{aligned} \tag{2.e}$$

3. Spectral theory for $-D^2$

This section contains an analysis of the generic structure of the spectrum for iD and $-D^2$. Constructive procedures are presented for recovering information about the graph $\mathcal{G}$ from the eigenvalues of $-D^2 = [iD]^2$. The information to be obtained consists of the real parts of the eigenvalues of the boundary matrix T and the lengths of loops in $\mathcal{G}$. Our initial focus is on the function

$$P(\lambda) = \det[\exp(-iW\lambda) - T]$$

whose zeroes are the eigenvalues of iD.

3.1. **Generic behaviour.** The justification of the inverse spectral procedures makes use of a number of assumptions about the operator $-D^2$. These assumptions include the following:

(i) each eigenvalue λ_k of iD has a corresponding eigenspace of dimension 1,

(ii) each eigenvalue λ_k of iD is a simple root of $P(\lambda)$.

For some results, arithmetic conditions on the weights w_n will be assumed. The various assumptions about the operator $-D^2$ are generically valid in senses which will be made precise.

For the first result the connectivity of $\mathcal{G}$ is ignored.

Proposition 3.1. *For every set of weights $\{w_n\}$ there is a unitary matrix T whose associated operator iD satisfies conditions (i) and (ii).*

Proof. The examples are found in the trivial case when our system is completely decoupled, the matrix T having the form

$$T = diag[\exp(i\theta_n)].$$

In this case eigenvalues of iD have the form

$$\theta_n/w_n + 2\pi k/w_n, \quad n = 1, \dots, N, \quad k = 0, \pm 1, \pm 2, \dots.$$

Condition (i) will be satisfied unless

$$\theta_m/w_m + 2\pi j/w_m = \theta_n/w_n + 2\pi k/w_n$$

or

$$\theta_m = \frac{w_m}{w_n}[\theta_n + 2\pi k] \mod 2\pi.$$

The values of θ_m may be chosen inductively. The first value $\theta_1 \in [0, 2\pi)$ is arbitrary. Subsequent choices are only excluded from a countable set of values, which shows that (i) may be satisfied.

If T has the specified form,

$$P(\lambda) = \prod_{n=1}^{N} [\exp(iw_n\lambda) - \exp(i\theta_n)].$$

If $\sigma_{m,k}$ is one of the eigenvalues λ_j of the form

$$\sigma_{m,k} = \theta_m/w_m + 2\pi k/w_m, \quad m = 1, \dots, N, \quad k = 0, \pm 1, \pm 2, \dots,$$

then

$$\partial_\lambda P(\sigma_{m,k}) = iw_m \exp(iw_n\sigma_{m,k}) \prod_{n \ne m} [\exp(iw_n\sigma_{m,k}) - \exp(i\theta_n)] \ne 0. \quad \square$$

It will also be important to observe that Proposition 3.1 can be satisfied with matrices $T(v)$ respecting the connectivity of $\mathcal{G}$.

Proposition 3.2. *For every set of weights $\{w_n\}$ there is a collection of unitary matrices*

$$T_0(v) : \mathcal{C}^{\delta(v)} \to \mathcal{C}^{\delta(v)}$$

whose associated operator iD satisfies conditions (i) and (ii).

Proof. This more restrictive case may be reduced to the previous one.

Start at any vertex and follow a directed edge path until a vertex v is repeated. Since each vertex of $\mathcal{G}$ has the same number of input and output edges, the edges of the path from v to v may be removed from $\mathcal{G}$, leaving another graph of the same type. Thus $\mathcal{G}$ may be decomposed into an edge disjoint collection of closed paths.

Let γ be one of these closed paths, and select one vertex v from γ. At each of the other vertices w of γ there is one entering edge e_i and one exiting edge e_j. Let T be partially defined by the condition $f_i(1) = f_j(0)$, that is

$$T_{ij} = 1, \quad T_{lj} = 0, \quad l \neq i.$$

These closed paths may now be considered intervals with weights $\sum w_m$, the sum taken over the edges in the path. The proof of Proposition 3.1 implies that values $\exp(i\theta_v)$ may be selected for boundary conditions at v giving the eigenvalue conditions (i) and (ii) as before. $\square$

Propositions 3.1 and 3.2 can be extended to generic operators of the form iD by using some results on analytic families of operators. Let T_0 and T_1 be unitary matrices with logarithms iA_0 and iA_1:

$$T_j = \exp(iA_j), \quad A_j = A_j^*, \quad j = 0, 1.$$

The real analytic path of unitary matrices

$$T(s) = \exp(i[(1-s)A_0 + sA_1]), \quad -\infty < s < \infty,$$

satisfies $T(0) = T_0$ and $T(1) = T_1$. Denote by iD_s the operator iD with domain defined by the boundary conditions using the unitary matrix $T(s)$. The corresponding resolvents are denoted $R_s(\lambda)$.

Lemma 3.3. *The eigenvalues $\lambda_k(s)$, appearing with the appropriate multiplicity, are well defined real analytic functions of s. There is also a corresponding complete orthonormal real analytic family $\{\psi_k(s)\}$ of eigenvectors for iD_s.*

Proof. From the explicit formula (2.e) we see that if $s \in (a, b)$ and λ is real and in the resolvent set of iD_s, then the operators $R_s(\lambda)$ form a self adjoint holomorphic family [12, p. 385] of compact operators. The eigenvalues $\lambda_k(s)$ of iD_s and the eigenvalues $\nu_k(s)$ of $R_s(\lambda)$ are related by

$$\nu_k(s) = \frac{1}{\lambda_k(s) - \lambda}. \tag{3.a}$$

For each $s_0 \in \mathcal{R}$ there are a real λ in the resolvent set $\rho(iD_{s_0})$ of iD_{s_0} and an interval (a, b) containing s_0 such that for all $s \in (a, b)$ we have $\lambda \in \rho(iD_s)$. Since the operators $R_s(\lambda)$ are resolvents, 0 is never an eigenvalue of $R_s(\lambda)$. The discussion in [12, p. 393] shows that each $\nu_k(s)$ is a real analytic function for $s \in (a, b)$. By (3.a) the same is true for $\lambda_k(s)$.

If $t_0, t_1 \in \mathcal{R}$ there is a finite open cover of $[t_0, t_1]$ by intervals (a_j, b_j) such that $(a_j, b_j) \cap (a_{j+1}, b_{j+1}) \neq \emptyset$ and such that the eigenvalues of iD_s are real analytic in (a_j, b_j). Now index the eigenvalues $\lambda_k(s)$ according to their index in the first interval (a_1, b_1) and extend these functions from (a_j, b_j) to (a_{j+1}, b_{j+1}) by requiring the definition to be consistent on $(a_j, b_j) \cap (a_{j+1}, b_{j+1})$.

The eigenvectors are handled analogously. $\square$

Suppose that $\mathcal{G}$ has M vertices. The $M-tuple$ of transition matrices defining the domain of iD is an element of the product of unitary groups $\mathcal{U} = \bigtimes_{m=1}^{M} U(\delta(v_m))$. The next result considers the size of the set in this product where conditions (i) and (ii) do not hold.

Theorem 3.4. *Suppose that the graph $\mathcal{G}$ has arbitrary weights $\{w_n\}$ with in and out degrees $\delta(v)$ at the vertices v. Then conditions (i) and (ii) hold except for a set of transition matrices having measure zero in $\mathcal{U} = \bigtimes_{m=1}^{M} U(\delta(v_m))$.*

Proof. The proof begins by showing that conditions (i) and (ii) hold at most points along certain curves in $\mathcal{U}$. According to Proposition 3.2 there is some choice of unitary transition matrices

$$T_0(v) : \mathcal{C}^{\delta(v)} \to \mathcal{C}^{\delta(v)}$$

satisfying conditions (i) and (ii). There are self adjoint matrices $A_0(v)$ and $A(v)$ such that

$$T_0(v) = \exp(iA_0(v)), \quad T(v) = \exp(iA(v)).$$

Define $T_s(v) = \exp(i[(1-s)A_0(v) + sA(v)])$.

By Lemma 3.3 the eigenvalues (with multiplicity) and orthonormal eigenvectors of iD_s may be chosen to be real analytic functions of s. Now condition (i) fails if and only if $\lambda_j(s) - \lambda_k(s) = 0$ for some $j \ne k$. Since none of these differences is 0 for all s, each may vanish only finitely many times between $s = 0$ and $s = 1$. Thus there are at most countably many points s where (i) does not hold.

A similar argument works for condition (ii). The function $\partial_\lambda P(\lambda_j(s))$ is analytic in s. At $s = 0$ we have $\partial_\lambda P(\lambda_j(0)) \ne 0$, so that $\partial_\lambda P(\lambda_j(s)) \ne 0$ for all j except at countably many values of s.

The remainder of the proof consists of extending the result obtained so far to a measure theoretic statement on $\mathcal{U}$. Only a sketch of the argument is provided.

First note that the self adjoint $\delta(v_m) \times \delta(v_m)$ matrices may be considered a complex Euclidean space by ignoring the entries below the diagonal. The exponential map $A \to \exp(iA)$ is a local diffeomorphism from the self adjoint matrices onto $U(\delta(v_m))$. Let K be a positive integer. It is possible to find a finite open covering $\{V_\alpha\}$ and numbers r_α with $K < r_\alpha < K+1$ such that $\pm r_\alpha$ are in the resolvent set for all operators iD defined by transition matrices in V_α. Since the set of transition matrices $\{T(v)\}$ for which (i) and (ii) hold is dense, we may take V_α to be the image, under the exponential map, of a Euclidean ball, and the center point of the ball gives an operator for which (i) and (ii) hold.

Using the ideas in the proof of Lemma 3.3, and in [12, pp. 109, 116], the eigenvalues $\lambda_1, \dots, \lambda_k$ with absolute value smaller than r_α are continuous functions in V_α. Two of these eigenvalues agree when $\prod_{i \ne j}(\lambda_i - \lambda_j) = 0$, and this zero set $E(\alpha, K)$ is measurable in V_α. To get the measure of this set, integrate its characteristic function in polar coordinates. By the first part of the proof the integral is zero along each radial line, so the set $E(\alpha, K)$ has measure 0. Finally, the set of unitary matrices for which condition (i) fails is $\bigcup_K \bigcup_\alpha E(\alpha, K)$, which has measure 0.

Again a similar argument works for condition (ii). $\square$

Having considered the generic behaviour of iD for $\mathcal{T} = (T(v_1), \dots, T(v_M)) \in \mathcal{U}$, we now derive a basic result for the general case.

Theorem 3.5. *Suppose that $\lambda_k(\mathcal{T})$ are the eigenvalues of iD, listed with multiplicity given by the dimension of the corresponding eigenspace. Let m be the dimension of the null space of iD. Then there are complex numbers β and C, with $C \neq 0$, such that*

$$P(\lambda) = C\lambda^m e^{\beta\lambda} \prod_{k=1}^{\infty} (1 - \lambda/\lambda_k) e^{\lambda/\lambda_k}.$$

In particular, the multiplicities given by the dimension of the eigenspaces agrees with the analytic multiplicities of λ_k as a root of $P(\lambda)$.

Proof. It will be necessary to have a coarse estimate for the distribution of eigenvalues for iD. If the boundary conditions defining the domain of iD were $f_n(0) = f_n(1)$ for $n = 1, \dots, N$, then a direct computation would show that the eigenvalues are $2\pi k/w_n$ for integer k and $n = 1, \dots, N$. Since the examples of interest differ from this trivial one only in the set of N boundary conditions, it is elementary ([2], Lemma 1.2) to show that the number of eigenvalues appearing in any open interval, counting with multiplicity, cannot change by more than N.

For notational simplicity assume that $m = 0$. The entire function $P(\lambda)$ has order 1, so by Hadamard's theorem [1, p. 207] its genus is either 0 or 1. Since $\sum 1/|\lambda_k| = \infty$ and $\sum 1/|\lambda_k|^2 < \infty$, the genus [1, p. 195] is 1 and $P(\lambda)$ has a product representation

$$P(\lambda) = Ce^{\beta\lambda} \prod_{k=1}^{\infty} (1 - \lambda/\sigma_k) e^{\lambda/\sigma_k}.$$

In this representation the set of numbers $\{\sigma_k\}$ is the same as the set $\{\lambda_k\}$, but the multiplicities might be different. According to Theorem 3.4 these multiplicities agree for a dense set in $\mathcal{U}$. We will write $\mathcal{T} \in \mathcal{U}$ as the limit of a sequence $\mathcal{T}_n \in \mathcal{U}$ satisfying (i) and (ii), and use a continuity argument to show that the multiplicities must agree for all $\mathcal{T}$.

From the definition

$$P(\lambda, \mathcal{T}) = \det[\exp(-iW\lambda) - T]$$

it is evident that $P(\lambda, \mathcal{T}_n) \to P(\lambda, \mathcal{T})$ uniformly on compact subsets of $\mathcal{C}$. It is a consequence of Rouche's theorem [1, p. 152] that each root σ of $P(\lambda)$ of multiplicity M is the limit of exactly M eigenvalues $\lambda_k(\mathcal{T}_n)$.

On the other hand, (2.e) shows that the resolvents for iD corresponding to $\mathcal{T}_n$ converge uniformly in operator norm to the resolvent corresponding to $\mathcal{T}$ on compact subsets of the resolvent set corresponding to $\mathcal{T}$. Suppose that $\lambda(\mathcal{T})$ is an eigenvalue of iD with eigenspace of dimension M. Then since iD is self adjoint, the eigenprojection [12, p. 181] associated to $\lambda(\mathcal{T})$ has rank M and [12, pp. 213-214] is the limit of exactly M eigenvalues $\lambda_k(\mathcal{T}_n)$.

From the observations of the previous two paragraphs, we see that the multiplicities given by the dimension of the eigenspaces agree with the analytic multiplicities of λ_k as a root of $P(\lambda)$. Consequently, we may take $\sigma_k = \lambda_k$ in the formula for $P(\lambda)$. $\square$

3.2. **Inverse problems.** Now we consider recovering graph information from the eigenvalues μ_k of operators $-D^2$. In the general case it will be necessary to assume that the multiplicities of the eigenvalues, that is the dimensions of the eigenspaces, are also known. However Theorem 3.4 shows that the generic case has all eigenvalues simple.

Of course having the values μ_k is the same as knowing the magnitudes $|\lambda_k|$ of the eigenvalues for iD. Although the signs of the eigenvalues are unknown, we do know that the set of values $\pm|\lambda_k|$ comprises the union of the spectra of $\pm iD$. By (2.d) these values are the roots of $\det[\exp(-iW\lambda) - T] = 0$ and $\det[\exp(iW\lambda) - T] = 0$. Since

$$\det[\exp(iW\lambda) - T] = \det[T]\det[T^{-1} - \exp(-iW\lambda)]\det[\exp(iW\lambda)], \tag{3.b}$$

this second equation may be replaced with the equivalent condition

$$\det[T^{-1} - \exp(-iW\lambda)] = 0.$$

Thus we consider the entire function

$$\begin{aligned} Q(\lambda) &= \det[\exp(-iW\lambda) - T][T^{-1} - \exp(-iW\lambda)] \\ &= -\det[I - T\exp(-iW\lambda) - \exp(-iW\lambda)T^{-1} + \exp(-2iW\lambda)]. \end{aligned}$$

Lemma 3.6. *The eigenvalues μ_k of $-D^2$ together with their multiplicities determine the function*

$$Q(\lambda) = \det[\exp(-iW\lambda) - T][T^{-1} - \exp(-iW\lambda)].$$

Proof. By Theorem 3.5 and (3.b), $Q(\lambda)$ has the product representation

(3.c)

$$Q(\lambda) = C\lambda^m e^{\alpha\lambda}\Big[\prod_{k=1}^{\infty}(1 - \lambda/|\lambda_k|)e^{\lambda/|\lambda_k|}\Big]\Big[\prod_{k=1}^{\infty}(1 + \lambda/|\lambda_k|)e^{-\lambda/|\lambda_k|}\Big], \quad |\lambda_k| \neq 0.$$

Let $Q_1(\lambda)$ be the entire function

$$Q_1(\lambda) = \lambda^m\Big[\prod_{k=1}^{\infty}(1 - \lambda/|\lambda_k|)e^{\lambda/|\lambda_k|}\Big]\Big[\prod_{k=1}^{\infty}(1 + \lambda/|\lambda_k|)e^{-\lambda/|\lambda_k|}\Big], \quad |\lambda_k| \neq 0,$$

constructed from the values μ_k.

Restrict λ to the imaginary axis, $\lambda = i\sigma$. From the form of $Q(\lambda)$ it follows that

$$\lim_{\sigma\to-\infty} Q(i\sigma) = -1. \tag{3.d}$$

From the product representation (3.c) and the limiting behaviour (3.d) it follows that there is a value of α such that

$$\lim_{\sigma\to-\infty} e^{i\alpha\sigma}Q_1(i\sigma)$$

has a nonzero complex value. It is easily seen that this choice of α is unique, so that the condition (3.d) determines α, and then C. ☐

Consider the case when the weights have a known common value $w_n = w$. Then Lemma 3.6 says we know the function

$$\begin{aligned} q(\lambda) &= -\det[I - T\exp(-iw\lambda) - \exp(-iw\lambda)T^{-1} + \exp(-2iw\lambda)I] \\ &= -\exp(-2iwN\lambda)\det[\exp(2iw\lambda)I - \exp(iw\lambda)T - T^{-1}\exp(iw\lambda) + I]. \end{aligned}$$

In this case $2Nw$ can be determined by the asymptotics as $\lambda \to i\infty$ and so N is determined. Construct

$$-\exp(iNw\lambda)q(\lambda)/2^N = \det[\cos(w\lambda)I - Re(T)].$$

Let the eigenvalues of T be $\phi_1, \dots, \phi_N$. By taking $\lambda = \cos^{-1}(x)/w$ for $-1 \le x \le 1$ we obtain the characteristic polynomial

$$\det[xI - Re(T)] = (x - Re(\phi_1)) \cdots (x - Re(\phi_N)).$$

This establishes the following result.

Theorem 3.7. *If the weights have a known common value $w_n = w$, then the eigenvalues μ_k of $-D^2$ together with their multiplicities determine N and the real parts of the eigenvalues ϕ_n of the unitary matrix T.*

The case of known distinct weights can be reduced to the previous case if a mild arithmetic condition [13, pp. 37–40] is satisfied.

Theorem 3.8. *Suppose that the eigenvalues μ_k of $-D^2$ together with their multiplicities are known. If in addition the weights w_n are given and the line*

$$\lambda(w_1, \dots, w_N) \mod 2\pi Z^N$$

is dense in the torus $R^N/[2\pi Z^N]$, then the real parts of the eigenvalues ϕ_n of the unitary matrix T may be determined.

Proof. Since the weights are known, Lemma 3.6 implies that the function

$$\begin{aligned} q_1(\lambda) &= -2^{-N}\det[\exp(iW\lambda/2)]Q(\lambda)\det[\exp(iW\lambda/2)] \\ &= \det[\cos(\lambda W)I - \exp(iW\lambda/2)T\exp(-iW\lambda/2)/2 \\ &\qquad - \exp(-iW\lambda/2)T^{-1}\exp(iW\lambda/2)/2] \end{aligned}$$

can be constructed. Pick a number $x \in [-1,1]$ and choose a sequence of points λ_r such that

$$\lambda_r(w_1, \dots, w_N) \to (\cos^{-1}(x), \dots, \cos^{-1}(x)) \mod 2\pi Z^N.$$

Then

$$q_1(\lambda_r) \to \det[xI - Re(T)].$$

Since x was arbitrary we again recover the characteristic polynomial of $Re(T)$. □

Finally we consider extracting information about the weights from the eigenvalues μ_k. This analysis begins with the forward problem of describing the function $P(\lambda) = \det[\exp(-iW\lambda) - T]$, assuming that we know the weights w_n. Recall that the matrix T has entries T_{ij} which can only be nonzero when i and j are respectively the indices of an input edge and an output edge at some vertex v. Following the terminology of topology rather than graph theory, define a cycle to be the sum of closed (directed) paths [1, p. 138].

Lemma 3.9. *Suppose that the diagonal terms T_{ii} are all zero. Then $P(\lambda)$ is a linear combination of exponentials*

$$P(\lambda) = \det[\exp(-iW\lambda) - T] = \sum_k c_k \exp(-i\lambda \sum_{n=1}^{N(k)} w_n)$$

where each nonzero sum of weights is taken over a cycle of at most N edges.

Proof. Let $a_{ij}(\lambda)$ denote the entries of the matrix $\exp(-iW\lambda) - T$. The function $P(\lambda)$ may be written as a sum of products, each summand having the form $S_\sigma(\lambda) = \mathrm{sgn}(\sigma)\prod a_{i\sigma(i)}$ where σ is a permutation of the indices $1,\dots,N$.

Select a vertex v with $\delta(v)$ input edges $m_1,\dots,m_\delta$ and output edges $n_1,\dots,n_\delta$. The condition $T_{ii} = 0$ means that no input edge index at v is also an output edge index at v. Let $S_\sigma(\lambda)$ be a nonzero summand whose permutation σ fixes exactly r of the input edge indices at v, i.e. $\sigma(m_k) = m_k$. Then $S_\sigma(\lambda)$ has the r factors $\exp(-iw_{m_k}\lambda)$.

Next we consider the columns whose indices n_l are output edge indices at v. Since $T_{in_l} = 0$ unless $i \in \{m_1,\dots,m_\delta\}$, and the r row indices m_k are already accounted for, there can be at most $\delta(v) - r$ factors a_{in_l} in S_σ with $i \in \{m_1,\dots,m_\delta\}$. The only other nonzero entries a_{ij} are on the diagonal, so that for at least r values of the output edge indices n_l, $\sigma(n_l) = n_l$.

The roles of input and output edges are interchangeable, so each product has an equal number of distinct diagonal contributions from inputs and outputs at each vertex. Of course each output edge at v is the input at another vertex. Decomposing $\mathcal{G}$ into edge disjoint loops as in the proof of Proposition 3.2 gives the result. $\square$

Rather than $P(\lambda)$, it is the function $Q(\lambda)$ which can be determined from the eigenvalues μ_k by Lemma 3.6. Recall that $\mathcal{G}$ has M vertices and $\mathcal{U} = \bigtimes_{m=1}^{M} U(\delta(v_m))$.

Theorem 3.10. *Suppose that the diagonal terms T_{ii} are all zero. Then $Q(\lambda)$ is a linear combination of exponentials*

$$Q(\lambda) = \det[\exp(-iW\lambda) - T][T^{-1} - \exp(-iW\lambda)] = \sum_k c_k \exp(-i\lambda \sum_{n=1}^{N(k)} w_n)$$

where each nonzero sum of weights is taken over a cycle of at most $2N$ edges. For an open dense set of unitary matrices $(T(v_1),\dots,T(v_M)) \in \mathcal{U}$, each frequency which is the length of a loop in $\mathcal{G}$ with no repeated edges appears with a nonzero coefficient c_k.

Proof. Since $T^{-1} = T^*$ Lemma 3.9 applies to both factors $\det[\exp(-iW\lambda) - T]$ and $\det[T^{-1} - \exp(-iW\lambda)]$ of $Q(\lambda)$. This shows immediately that $Q(\lambda)$ is the desired linear combination of exponentials.

To describe the generic behaviour we first consider some special examples. Partition $\mathcal{G}$ into edge disjoint loops with no repeated edges. If v is a vertex in one of the loops, with input edge i and output edge j, let T be defined by requiring $f_i(1) = f_j(0)$, except at the initial vertex where the condition will be $f_i(1) = -f_j(0)$. In this case we may reindex the edges so that $P(\lambda)$ is the determinant of a block diagonal matrix with blocks of the form

$$\begin{pmatrix} \exp(-iw_1\lambda) & -1 & 0 & 0 & \dots & 0 \\ 0 & \exp(-iw_2\lambda) & -1 & 0 & \dots & 0 \\ 0 & 0 & \vdots & \vdots & \dots & 0 \\ 0 & 0 & 0 & \dots & \exp(-iw_{m-1}\lambda) & -1 \\ 1 & 0 & 0 & 0 & \dots & \exp(-iw_m\lambda) \end{pmatrix}.$$

Thus after the reindexing $P(\lambda)$ is a product of terms of the form

$$\exp(-i\lambda \sum w_n) + (-1)^{m-1}(-1)^{m-1} = \exp(-i\lambda \sum w_n) + 1.$$

The second factor $\det[T^{-1} - \exp(-iW\lambda)]$ of $Q(\lambda)$ is, except for a factor -1, a product of terms of the same form. For this example each frequency which is the length of a loop in the partition of $\mathcal{G}$ with no repeated edges appears with a nonzero coefficient.

In the general case the coefficient c_k of a term $\exp(-i\lambda \sum_n w_n)$ with frequency $\sum_n w_n$ is a polynomial in the entries of the matrices $T(v_m)$ and $T^*(v_m)$. As in Theorem 3.4, local coordinates for $U(\delta(v))$ may be chosen by using the exponential map $A \to \exp(iA)$ where $A = A^*$. In these coordinates the real and imaginary parts of the coefficient c_k are real analytic functions of the real and imaginary parts of the entries of the matrices A. Our examples show that each loop length with no repeating edges has an associated nonzero coefficient c_k for some choice of $(T(v_1), \dots, T(v_M))$. By the real analyticity, this particular coefficient must be nonzero on an open dense set in $\mathcal{U}$. But then all coefficients coming from lengths of loops without repeating edges are nonzero on the intersection of finitely many open dense sets, which completes the proof. $\square$

If λ is restricted to real values, $Q(\lambda)$ is almost periodic. The distinct frequencies $\sum_{n=1}^{N(k)} w_n$ and the corresponding nonzero coefficients may be identified by considering the integrals [13, p. 14]

$$\frac{1}{2T} \int_{-T}^{T} Q(\lambda) \exp(i\nu\lambda)\ d\lambda.$$

4. The Zeta Function for Schrödinger's Operator

In this section attention shifts from $-D^2$ to the operator $-D^2 - p$, where p is a real valued function on $\mathcal{G}$ with several continuous derivatives. This operator has a zeta function $\zeta_p(s)$ much as in the smooth manifold case [18]. In the graph setting the resolvent of $-D^2$ can be exactly computed. Up to negligible corrections, all of the boundary conditions we have considered lead to the same diagonal for the kernel of the resolvent. This observation means that the singular part of $\zeta_p(s)$ can be computed by considering boundary conditions for which the graph reduces to edge disjoint loops. The local formulas for the residues of $\zeta_p(s)$ in such a periodic case have been known for a long time [8]. A well-known argument [14, p. 226] shows that the isospectral classes are compact.

4.1. **The resolvent of $-D^2$.** The goal for this section is to establish the following description of the resolvent of $-D^2$. Recall that $M(\lambda) = [I - \exp(i\lambda W)T]^{-1}$. For notational convenience, let $\lambda = -\omega^2$.

Lemma 4.1. *The resolvent $R(\lambda)$ of $-D^2$ is an integral operator satisfying*

$$\begin{aligned}
&[R(\lambda)f](x) \qquad\qquad (4.a)\\
&= -\frac{1}{2\omega} \int_0^1 \exp(-x\omega W) M(-i\omega)[\exp(2\omega W) - I] M(i\omega) \exp(-\omega t W) W f(t)\ dt\\
&+\frac{1}{2\omega} \int_0^1 \exp(-x\omega W) M(-i\omega)[\exp(2\omega W) - \exp(2t\omega W)] \exp(-\omega t W) W f(t)\ dt\\
&+\frac{1}{2\omega} \int_0^1 \exp(-\omega x W)[\exp(2x\omega W) - I] M(i\omega) \exp(-\omega t W) W f(t)\ dt\\
&-\frac{1}{2\omega} \int_0^x \exp(-\omega x W)[\exp(2x\omega W) - \exp(2t\omega W)] \exp(-t\omega W) W f(t)\ dt.
\end{aligned}$$

For all $j,k = 0,1,2,\dots$ and $\epsilon > 0$ the diagonal of the resolvent kernel has the following behaviour as $\omega \to \infty$:

$$\partial_x^j \partial_\lambda^k R(x,x,\lambda) = \partial_x^j \partial_\lambda^k \frac{1}{2\omega} W + O(\exp(-\min_n [w_n/2 - \epsilon]\omega)).$$

Proof. Using (2.e) and

$$R(-\omega^2) = [-D^2 + \omega^2]^{-1} = (iD + i\omega)^{-1}(iD - i\omega)^{-1}$$

leads to

$$[R(\lambda)f](x) = [R_1(-i\omega)R_1(i\omega)f](x)$$
$$= \exp(-x\omega W)M(-i\omega) \int_0^1 i\exp(s\omega W)W$$
$$\times \Big[\exp(s\omega W)M(i\omega)\int_0^1 i\exp(-t\omega W)Wf(t)\,dt - \int_0^s i\exp(-[t-s]\omega W)Wf(t)\,dt\Big]\,ds$$
$$-\int_0^x i\exp([s-x]\omega W)W$$
$$\times \Big[\exp(s\omega W)M(i\omega)\int_0^1 i\exp(-t\omega W)Wf(t)\,dt - \int_0^s i\exp(-[t-s]\omega W)Wf(t)\,dt\Big]\,ds.$$

Interchanging the orders of integration gives (4.a).

After a bit of manipulation, the resolvent kernel for $t \le x$, is

$$\begin{aligned} &R(x,t,\lambda)\\ &= \frac{1}{2\omega}\exp(-x\omega W)M(-i\omega)\exp(2\omega W)[I - M(i\omega)]\exp(-t\omega W)W\\ &\quad + \frac{1}{2\omega}\exp(-x\omega W)[M(-i\omega) - I]M(i\omega)\exp(-t\omega W)W\\ &\quad + \frac{1}{2\omega}\exp(-x\omega W)[I - M(-i\omega)]\exp(t\omega W)W\\ &\quad + \frac{1}{2\omega}\exp(x\omega W)[M(i\omega) - I]\exp(-t\omega W)W. \end{aligned}$$

The functions

$$M(\pm i\omega) = [I - \exp(\mp\omega W)T]^{-1}$$

have series representations valid as $\omega \to \infty$ (or $\lambda \to -\infty$). These are

$$M(i\omega) = \sum_{k=0}^{\infty}\Big(\exp(-\omega W)T\Big)^k \simeq I + \exp(-\omega W)T,$$

$$M(-i\omega) = -T^{-1}\exp(-\omega W)\sum_{k=0}^{\infty}\Big(T^{-1}\exp(-\omega W)\Big)^k \simeq -T^{-1}\exp(-\omega W).$$

Inserting these series into the expression for the diagonal of the resolvent kernel gives

$$2\omega R(x,x,-\omega^2)W^{-1} = J_1 + \cdots + J_4 + E(x,\omega),$$

where the J_j use terms up to $k = 1$ in the series and E represents the remainder. More explicitly,

$$J_1 = \exp(-2x\omega W), \quad J_2 = -\exp(-2x\omega W) + E_1(x,\omega),$$

$$J_3 = I + \exp(-x\omega W)T^{-1}\exp(-[1-x]\omega W), \quad J_4 = \exp(-[1-x]\omega W)T\exp(-x\omega W).$$

Notice that J_4, and $J_3 - I$, and all of their derivatives with respect to x and ω, are $O(\exp(-min_n w_n/2\omega))$ as $\omega \to \infty$. The same is true for $E(x,\omega)$ and $E_1(x,\omega)$. □

4.2. **The trace of $R_p(\lambda)$.** If p is a bounded real valued function on $\mathcal{G}$, then the operator $-D^2 - p$ is self adjoint [12, p. 287] on the domain of $-D^2$. Its resolvent will be denoted by

$$R_p(\lambda) = [-D^2 - p - \lambda]^{-1}.$$

These resolvents may be represented as $N \times N$ matrices of integral operators,

$$R_p(\lambda)f = \begin{pmatrix} \sum_{m=1}^N \int_0^1 R_{m1}(x,t,\lambda) f_m(t)\ dt \\ \vdots \\ \sum_{m=1}^N \int_0^1 R_{mN}(x,t,\lambda) f_m(t)\ dt \end{pmatrix},$$

where the matrix entries $R_{mn}(x,t,\lambda)$ are continuous functions of $(x,t) \in [0,1] \times [0,1]$ for λ in the resolvent set. This result may be developed by mimicking the classical proof [5, p. 192] in case $N = 1$. An alternative is to first consider the decoupled problem with periodic boundary conditions $f_n(0) = f_n(1)$. In this case the resolvent may be represented as a diagonal matrix of integral operators, and the change of boundary conditions has a minor effect which may be analyzed as in [12, p. 188].

The distribution of eigenvalues μ_k of $-D^2 - p$ will look roughly like that of the squares of integers k^2. Let $\mathcal{N}(\lambda)$ be the eigenvalue counting function

$$\mathcal{N}(\lambda) = \#\{\mu_k \le \lambda\}.$$

The counting function will satisfy the coarse estimate [2, p. 174]

$$c_1 \lambda^{1/2} \le \mathcal{N}(\lambda) \le c_2 \lambda^{1/2}, \quad \lambda > 1, \quad c_1, c_2 > 0,$$

which suffices for our needs.

Lemma 4.2. *The eigenfunctions $\phi_k(x)$ of $-D^2 - p$ satisfy an estimate*

$$|\phi_{k,n}(x)| \le C \|\phi_k\|_2, \quad \phi_k = \begin{pmatrix} \phi_{k,1} \\ \vdots \\ \phi_{k,N} \end{pmatrix},$$

where C is independent of k.

Proof. The components of eigenfunctions satisfy the equations

$$-\phi_{k,n}''(x) - w_n^2 p_n(x) \phi_{k,n}(x) = w_n^2 \mu_k \phi_{k,n}(x), \quad x \in [0,1].$$

Each component may be written as a linear combination

$$\phi_{k,n}(x) = a y_1(x,\mu_k) + b y_2(x,\mu_k)$$

where the y_j satisfy the same equations and have the initial data

$$y_j^{(i-1)} = \delta_{ij}, \quad i,j = 1,2.$$

When $p = 0$ the solution with the same initial data is

$$a \cos(w_n \sqrt{\mu_k} x) + b \frac{\sin(w_n \sqrt{\mu_k} x)}{w_n \sqrt{\mu_k}} = c \cos(w_n \sqrt{\mu_k} x - \alpha),$$

where

$$c^2 = a^2 + \frac{b^2}{w_n^2 |\mu_k|}.$$

Thus

$$\int_0^1 |b\frac{\sin(w_n\sqrt{\mu_k}x)}{w_n\sqrt{\mu_k}} + a\cos(w_n\sqrt{\mu_k}x)|^2\ dx$$

$$= \Big[a^2 + \frac{b^2}{w_n^2|\mu_k|}\Big]\Big[\frac{1}{2} + \frac{1}{4w_n\sqrt{\mu_k}}\sin(2w_n\sqrt{\mu_k}x - \alpha)\Big] \to \frac{1}{2}\Big[a^2 + \frac{b^2}{w_n^2|\mu_k|}\Big], \quad \mu_k \to \infty.$$

We also have the standard estimates [16, p. 13]

$$|y_1(x,\mu_k) - \cos(w_n\sqrt{\mu_k}x)| \le \frac{C}{w_n\sqrt{\mu_k}}, \tag{4.b}$$

$$|y_2(x,\mu_k) - \frac{\sin(w_n\sqrt{\mu_k}x)}{w_n\sqrt{\mu_k}}| \le \frac{C}{w_n^2\mu_k},$$

for some constant C independent of k and $\mu_k \ge 1$.

If $\|\cdot\|$ denotes the L^2 norm on $[0,1]$, then for μ_k sufficiently large an application of the triangle inequality yields

$$\|ay_1(x,\mu_k) + by_2(x,\mu_k)\| \ge \frac{1}{2}(a^2 + \frac{b^2}{w_n^2\mu_k})^{1/2} - C(\frac{|a|}{w_n\sqrt{\mu_k}} + \frac{|b|}{w_n^2\mu_k}).$$

This inequality shows that

$$\max(a, \frac{b}{w_n\sqrt{\mu_k}}) \le C\|ay_1(x,\mu_k) + by_2(x,\mu_k)\|.$$

Another application of the estimates (4.b) gives the result. □

The resolvents and certain other functions of $-D^2 - p$ will belong to the trace class (see [17, pp. 206–212], [12, pp. 523–525] or [18, pp. 249–260]). For operators A in the trace class, the trace is defined by

$$\sum_k (A\phi_k, \phi_k)$$

for any orthonormal basis $\{\phi_k\}$. The trace norm is $\|A\|_1 = \mathrm{tr}([A^*A]^{1/2})$. The bounds on eigenfunctions of $-D^2 - p$ can be used to justify a more convenient representation of the trace. This result, presented in the next lemma, is well known in similar contexts [18, p. 259].

Lemma 4.3. *Suppose that a function $h : \sigma(-D^2 - p) \to \mathcal{C}$ satisfies $\sum_k |h(\mu_k)| < \infty$. The operator*

$$A = h(-D^2 - p) : \bigoplus_n L^2([0,1], w_n) \to \bigoplus_n L^2([0,1], w_n)$$

is trace class, and may be represented as a matrix integral operator with kernel $A_{m,n}(x,t)$. The functions $A_{m,n}(x,t)$ may be chosen continuous in (x,t), and with this choice

$$\mathrm{tr}(A) = \sum_{n=1}^N \int_0^1 A_{n,n}(x,x)\ dx.$$

Proof. Use an orthonormal basis of eigenfunctions ϕ_k to write $f \in \bigoplus_n L^2([0,1], w_n)$ as $f = \sum_k a_k\phi_k$. Since $Af = \sum_k a_k h(\mu_k)\phi_k$,

$$Af(x) = \int_0^1 \begin{pmatrix} \sum_{n=1}^N A_{1,n}(x,t)f_n(t) \\ \vdots \\ \sum_{n=1}^N A_{N,n}(x,t)f_n(t) \end{pmatrix}\ dt,$$

where the kernel of A has the representation

$$A(x,t) = \Big(A_{m,n}(x,t)\Big), \quad A_{m,n}(x,t) = \sum_k h(\mu_k)\phi_{k,m}(x)\overline{w_n\phi_{k,n}(t)}. \tag{4.c}$$

By virtue of our assumptions on h and the bounds on orthonormal eigenfunctions in Lemma 4.2, the series in (4.c) converges uniformly to a continuous function. Note in particular that the coarse eigenvalue estimates for $-D^2 - p$ imply that the resolvents $R_p(\lambda)$ are in this class.

The trace takes the form

$$\sum_k (A\phi_k, \phi_k) = \sum_k \sum_{n=1}^N \int_0^1 (A\phi_k)_n(x)\overline{\phi_{k,n}(x)} w_n \ dx$$

$$= \sum_k \sum_{n=1}^N \int_0^1 h(\mu_k)\phi_{k,n}(x)\overline{\phi_{k,n}(x)} w_n \ dx = \sum_{n=1}^N \int_0^1 A_{n,n}(x,x) \ dx. \quad \square$$

To proceed further it will be convenient to assume that derivatives $p^{(j)}$ of the function p are continuous on the graph $\mathcal{G}$. This means that the various components p_n of p have j continuous derivatives on $(0,1)$, that these derivatives extend continuously to the closed interval $[0,1]$, and that at the endpoints

$$\lim_{x\to 0}[\frac{1}{w_n}\partial_x]^j p_n(x) = \lim_{x\to 1}[\frac{1}{w_m}\partial_x]^j p_m(x)$$

for all edges e_m, e_n respectively entering and exiting a common vertex. Continuity of enough derivatives $p^{(j)}$ on the graph, and particularly across the vertices, will imply that multiplication by p leaves the domain of $-D^2$ invariant. This will result in a simple commutation formula (4.j) for p and the resolvent $R(\lambda)$. The commutation formula will lead to an expansion for $R_p(\lambda)$ and expressions for the residues of the zeta function $\zeta_p(s)$ in which the matrix T does not appear. The symbol $H_{n,k}(p)$ will denote a polynomial in derivatives of p.

Lemma 4.4. *Suppose that $p^{(j)}$ extends continuously to the graph for $j \le \mathcal{J}$, $\mathcal{J} \ge 2$. Then for $Re(\lambda) < -\|p\|_\infty$,*

$$R_p(\lambda) = \sum_{n=2}^{\mathcal{J}} \Big[\sum_{k=\lceil n/2 \rceil}^{2n} H_{n,k}(p) D^{2k-n} R^k \Big] + E_{\mathcal{J}},$$

where the error $E_{\mathcal{J}}$ satisfies the estimates

$$\|E_{\mathcal{J}}\| = O(\omega^{-\mathcal{J}-4}), \quad \|E_{\mathcal{J}}\|_1 = O(\omega^{-\mathcal{J}-3}), \quad \omega^2 = -\lambda.$$

The polynomials $H_{n,k}(p)$ are the same for all unitary matrices $T(v)$ defining the domain of iD.

Proof. Since p is bounded, the perturbation series

$$R_p(\lambda) = R\sum_{n=0}^{\infty} [pR]^n = R + RpR + RpRpR + \dots$$

will converge if the distance from λ to the spectrum of $-D^2$ exceeds $\|p\|_\infty$.

For the nth edge the product rule gives

$$(-\partial_x^2/w_n^2 - \lambda)p_n f = p_n(-\partial_x^2/w_n^2 - \lambda)f - 2\frac{\partial_x p_n}{w_n^2}\partial_x f - \frac{\partial_x^2 p_n}{w_n^2} f.$$

If $f = R(\lambda)h$ where h is continuous on each edge, then

$$(-\partial_x^2/w_n^2 - \lambda)p_n R(\lambda)h = \Big[p_n(-\partial_x^2/w_n^2 - \lambda) - 2\frac{\partial_x p_n}{w_n^2}\partial_x - \frac{\partial_x^2 p_n}{w_n^2}\Big]R(\lambda)h$$

$$= p_n h - \Big[2\frac{\partial_x p_n}{w_n^2}\partial_x + \frac{\partial_x^2 p_n}{w_n^2}\Big]R(\lambda)h.$$

This equation extends by continuity to all $h \in L^2(\mathcal{G})$.

If $p^{(j)}$, $j = 0, 1, 2$, extend continuously across the vertices, multiplication by p maps the domain of $-D^2$ into itself, so that

$$R(\lambda)p_n = p_n R(\lambda) + 2R(\lambda)\frac{\partial_x p_n}{w_n^2}\partial_x R(\lambda) + R(\lambda)\frac{\partial_x^2 p_n}{w_n^2}R(\lambda).$$

Interpreting these equations globally on the graph, we will write

$$R(\lambda)p = pR(\lambda) + 2R(\lambda)p' DR(\lambda) + R(\lambda)p'' R(\lambda), \quad p' = Dp. \tag{4.d}$$

Now start with the identity

$$R_p(\lambda) = R[\sum_{n=0}^{\mathcal{J}-1}[pR]^n + (pR)^{\mathcal{J}}(1 - pR)^{-1}].$$

Since multiplication by p leaves the domain of $[-D^2]^k$ invariant, the term

$$R(pR)^{\mathcal{J}}(1 - pR)^{-1}$$

maps $L^2(\mathcal{G})$ into the domain of $[-D^2]^{\mathcal{J}+1}$ and satisfies the norm estimate

$$\|R(pR)^{\mathcal{J}}(1 - pR)^{-1}\| \le K_{\mathcal{J}}|\lambda|^{-\mathcal{J}-1}, \quad \lambda < 0.$$

Thanks to the trace norm estimate ([17, p. 218] or [18, p. 256])

$$\|AB\|_1 \le \|A\|_1\|B\|$$

and Lemmas 4.1 and 4.3,

$$\|R(pR)^{\mathcal{J}}(1 - pR)^{-1}\|_1 \le K_{\mathcal{J}}|\lambda|^{-\mathcal{J}}\|R\|_1 \le K_{\mathcal{J}}|\lambda|^{-\mathcal{J}-1/2}, \quad \lambda < 0.$$

Note that an operator of the form $D^j R^k$ with $j + 1 < 2k$ will map $L^2(\mathcal{G})$ into the domain of D^{2k-j} with the norm estimate

$$\|D^j R^k\| \le \omega^{2k-j}, \quad \omega^2 = -\lambda > 0,$$

and the trace norm estimate

$$\|D^j R^k\|_1 \le \omega^{2k-j-1}.$$

Thus the commutation formula (4.d) allows us to rewrite $R\sum_{n=0}^{\mathcal{J}-1}[pR]^n$ by pushing powers of R to the right until the desired form,

$$R_p(\lambda) = \sum_{n=2}^{\mathcal{J}}\Big[\sum_{k=\lceil n/2\rceil}^{2n} H_{n,k}(p)D^{2k-n}R^k\Big] + E_{\mathcal{J}},$$

and error estimates are achieved. Finally, the unitary matrices $T(v)$ defining the domain of iD did not enter into the generation of the polynomials $H_{n,k}(p)$. $\square$

4.3. **The zeta function for $-D^2 - p$.** Having established Lemmas 4.1, 4.3, and 4.4, it is now possible to obtain a description of the zeta function $\zeta_p(s)$ associated to a Schrödinger operator $L = -D^2 - p$ on a graph $\mathcal{G}$. For completeness we briefly sketch the development. A systematic treatment for the case of a manifold, which is quite similar, may be found in [18, pp. 82 – 114].

On one hand, $\zeta_p(s)$ has the simple description

$$\zeta_p(s) = \sum_k \mu_k^s, \quad Re(s) < -1/2,$$

where $\{\mu_k\}$ is the sequence of (nonzero) eigenvalues of L, taken with multiplicity. This function is the same as $\mathrm{tr}(L^s)$, which has an alternate contour integral description.

Theorem 4.5. *Suppose that $p^{(i)}$ extends continuously to the graph for $i \le \mathcal{J}$, $\mathcal{J} \ge 2$. The function $\zeta_p(s)$ has a meromorphic extension to $Re(s) < (\mathcal{J}+1)/2$, with poles located at the points $s = (2j-1)/2$ for $j = 0, 1, 2, \dots$. The poles are all simple, and the residues have the form*

$$\sum_{n=1}^{\mathcal{J}} \int_0^1 \mathcal{H}_{j,n}(p_n) \; dx,$$

where $\mathcal{H}_{j,n}(p)$ is a polynomial in p_n and its derivatives. The residues are independent of the particular choice of unitary matrices $T(v) : \mathcal{C}^{\delta(v)} \to \mathcal{C}^{\delta(v)}$.

Sketch of Proof. Slit the complex plane along the negative real axis, so that $\log(\lambda)$ is analytic in the complement of the slit, and consider the contour $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$ which contains the slit, where

$$\Gamma_1 = te^{i\pi}, \quad \infty > t > r, \quad \Gamma_3 = te^{-i\pi}, \quad r < t < \infty, \quad \Gamma_2 = re^{-i\theta}, \quad -\pi \le \theta \le \pi.$$

If all eigenvalues of L are greater than r, then (taking the logarithm real for $\lambda > 0$)

$$L^s = \frac{1}{2\pi i} \int_\Gamma \lambda^s R(\lambda) \; d\lambda = \frac{1}{2\pi i} \int_\Gamma e^{s \log(\lambda)} R(\lambda) \; d\lambda, \quad Re(s) < 0,$$

and

$$\mathrm{tr}(L^s) = \frac{1}{2\pi i} \int_\Gamma \lambda^s \, \mathrm{tr}(R(\lambda)) \; d\lambda. \tag{4.e}$$

If L has nonpositive eigenvalues, choose r so that Γ_2 lies in the resolvent set of L and encloses the nonpositive eigenvalues. Then the contour integral (4.e) will give

$$\sum_{\mu_k > r} \mu_k^s,$$

which differs from $\zeta_p(s)$ by an entire function.

The expansion of $R_p(\lambda)$ from Lemma 4.4. is now inserted into (4.e). The estimate $\|E_{\mathcal{J}}\|_1 = O(\omega^{-\mathcal{J}-3})$ implies that

$$\frac{1}{2\pi i} \int_\Gamma \lambda^s \, \mathrm{tr}(E_{\mathcal{J}}) \; d\lambda$$

defines an analytic function of s for $Re(s) < (\mathcal{J}+1)/2$. The remaining terms have the form

$$\frac{1}{2\pi i} \int_\Gamma \lambda^s \, \mathrm{tr}(H_{n,k}(p) D^{2k-n} R^k) \; d\lambda. \tag{4.f}$$

The powers of the resolvent satisfy

$$R(\lambda)^{j+1} = \frac{1}{2\pi i}\frac{1}{j!}\partial_\lambda^j R(\lambda),$$

so that by Lemma 4.1 the only terms from (4.f) contributing to the singularities of $\zeta_p(s)$ are those with no positive powers of D.

By Lemmas 4.1 and 4.3 the significant terms from (4.f) may be rewritten as

$$\Big[\sum_n \int_0^1 w_n H_{n,n/2}(p)\ dx\Big]\frac{1}{2\pi i}\int_\Gamma \lambda^s \frac{1}{(k-1)!}\partial_\lambda^{k-1}(2\omega)^{-1}\ d\lambda.$$

One calculates that the singular part of the contour integral is

$$\frac{1\cdot 3\cdots(2k-3)}{2^k(k-1)!}\frac{\sin(\pi s)}{\pi}\frac{-1}{s-[2k-3]/2}, \quad s<[2k-3]/2.$$

This establishes the existence of a meromorphic continuation for $\zeta_p(s)$, and the description of the singular part.

Finally, notice that the particular unitary vertex matrices $T(v)$ played no role in the description of the singular part of $\zeta_p(s)$. □

One may use the method of proof to calculate the residues, although the computations quickly become tedious. The first few residues are recorded in the following table.

pole	residue
$s=-1/2$	$\sum_n w_n/[2\pi]$
$s=1/2$	$-[\sum_n(\int_0^1 w_n p_n)]/[4\pi]$
$s=3/2$	$3[\sum_n(\int_0^1 w_n p_n^2)]/[16\pi]$
$s=5/2$	$-5[\sum_n(\int_0^1 w_n[p_n^3-(1/2)(\partial_x p_n/w_n)^2])]/[32\pi]$

.

A simpler approach is to make a judicious choice of the matrices $T(v)$. Partition $\mathcal{G}$ into edge disjoint loops with no repeated edges, as in Proposition 3.2. If v is a vertex in one of the loops, with input edge i and output edge j, let T be defined by requiring $f_i(1)=f_j(0)$. In this case after a change of coordinates the operator $-D^2-p$ may be recognized as a diagonal operator

$$\frac{1}{2\pi}\begin{pmatrix} -\partial_s^2-\tilde{p}_1 & 0 & \dots & 0 \\ 0 & -\partial_s^2-\tilde{p}_2 & \dots & 0 \\ 0 & 0 & \dots & -\partial_s^2-\tilde{p}_M \end{pmatrix}$$

acting on $\bigoplus_m L^2(S^1_{r(m)})$, where $S^1_{r(m)}$ denotes the circle of radius $r(m)$ and s is arc length. Thus residue computations may take advantage of the previously developed art [8, 14].

Our main interest in the development of $\zeta_p(s)$ was to see that the set of operators $-D^2-p$ having the same eigenvalues is compact. The argument is familiar. If the eigenvalues are given, then $\zeta_p(s)$ is determined, and in particular the residues are fixed. One shows from the explicit form of the residues that both p and its derivative are bounded in L^2. The details of a more general version of this argument may be found in [14, p. 226], where the equivalent heat kernel invariants [11, p. 56], [18, pp. 113–114] are used. Thus the following result is obtained.

Corollary 4.6. *Consider potentials p, q with at least 5 continuous derivatives on $\mathcal{G}$. The isospectral sets*

$$M_p = \{q \mid \mu_k(q) = \mu_k(p), \quad k = 1, 2, 3, \dots\}$$

are compact in $\bigoplus_n L^2([0,1], w_n)$.

References

1. L. Ahlfors, *Complex analysis*, McGraw-Hill, New York, 1966. MR **32:**5844
2. R. Carlson, *Expansions associated with non-self-adjoint boundary-value problems*, Proceedings of the American Mathematical Society **73** (1979), no. 2, 173–179. MR **80e:**47040
3. F. Chung, *Spectral graph theory*, American Mathematical Society, 1997. MR **97k:**58183
4. F. Chung and S. Sternberg, *Laplacian and vibrational spectra for homogeneous graphs*, Journal of Graph Theory **16** (1992), no. 6, 605–627. MR **93j:**58135
5. E.A. Coddington and N. Levinson, *Theory of ordinary differential equations*, McGraw-Hill, New York, 1955. MR **16:**1022b
6. E. Davies, *Large deviations for heat kernels on graphs*, Journal of the London Mathematical Society (2) **47** (1993), 65–72. MR **94f:**58135
7. Y. Colin de Verdiere, *Spectre du Laplacien et longueurs des geodesiques periodiques II*, Compositio Mathematica **27** (1973), no. 2, 159–184. MR **50:**1293
8. L. Dikii, *Trace formulas for Sturm-Liouville operators*, American Mathematical Society Translations **18** (1958), 81–115. MR **23:**A1874
9. P. Exner and P. Seba, *Schroedinger operators on unusual manifolds*, Ideas and methods in quantum and statistical physics (Oslo 1988) (S. Albeverio, J. Fenstad, H. Holden, and T. Lindstrom, eds.), 1992, pp. 227–253. MR **94a:**81026
10. N. Gerasimenko and B. Pavlov, *Scattering problems on noncompact graphs*, Theoretical and Mathematical Physics **74** (1988), no. 3, 230–240. MR **90f:**47010
11. P. Gilkey, *The index theorem and the heat equation*, Publish or Perish, Boston, 1974. MR **56:**16704
12. T. Kato, *Perturbation theory for linear operators*, Springer, New York, 1995. MR **96a:**47025
13. B.M. Levitan and V.V. Zhikov, *Almost periodic functions and differential equations*, Cambridge University Press, Cambridge, 1982. MR **84g:**34004
14. H. McKean and P. van Moerbeke, *The spectrum of Hill's equation*, Inventiones Math. **30** (1975), 217–274. MR **53:**936
15. B. Mohar and W. Woess, *A survey on spectra of infinite graphs*, Bull. London Math. Soc. **21** (1989), 209–234. MR **90d:**05162
16. J. Poschel and E. Trubowitz, *Inverse spectral theory*, Academic Press, Orlando, 1987. MR **89b:**34061
17. M. Reed and B. Simon, *Methods of modern mathematical physics, 1*, Academic Press, New York, 1972. MR **58:**12429a
18. M. Shubin, *Pseudodifferential operators and spectral theory*, Springer-Verlag, Berlin, 1987. MR **88c:**47105
19. P. Sy and T. Sunada, *Discrete Schrödinger operators on a graph*, Nagoya Math Journal **125** (1992), 141–150. MR **93c:**58232

Department of Mathematics, University of Colorado at Colorado Springs, Colorado Springs, Colorado 80933

E-mail address: carlson@vision.uccs.edu